Dromion Interactions of a (2+1)-Dimensional sine-Gordon Equation

Hang-yu Ruan [a,b] and Yi-xin Chen [a]

[a] Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou, 310027, China
[b] Institute of Modern Physics, Normal College of Ningbo University, Ningbo, 315211, China

Reprint requests to H.-y. R.

Z. Naturforsch. **54a,** 291–298 (1999); received December 22, 1998

Starting from two line solitons, the solution of an integrable (2+1)-dimensional sine-Gordon equation in bilinear form yields a dromion solution that is localized in all directions for a suitable potential. The interaction between two dromions is studied in detail through the method of figure analysis. For different selections of parameters, the interactions between two dromions may be elastic, not completely elastic, or completely inelastic.

1. Introduction

Since the pioneering work of Boiti, Leon, Martina and Pempinelli [1], the study of exponentially localized soliton solutions, called dromions, in (2+1) dimensions has attracted much attention of physicists and mathematicians. Usually, dromion solutions are driven by two or more nonparallel straight-line ghost solitons. For instance, for the Davey-Stewartson (DS) [2] and the Nizhnik-Novikov-Veselov (NNV) [3] equations, their dromion solutions are driven by two perpendicular line ghost solitons [1, 4]. For the Kadomtsev-Petviashvili (KP) equation, the dromion solutions are driven by nonperpendicular line ghost solitons [5]. For one type of nonlinear models, say the DS, NNV, and asymmetrical NNV (ANNV) [6] equations, dromion solutions exist for physical fields. However, for other types of equations like the KP and the breaking soliton equations, dromion solutions exist only for some suitable potentials of the field [5, 7]. More recently, even more generalized dromion solutions which are driven by curved and straight line solitons for some types of (2+1)-dimensional nonlinear modes were found [8].

In this paper we are interested in the interaction of dromions for the (2+1)-dimensional integrable sine-Gordon equation (SGE). The SGE plays a central role in such diverse fields as differential geometry [9], nonlinear optics [10], plasma physics [11], superconductivity [12], and particle physics [13]. The present popularity of the SGE in particle physics is primarily due to its intimate connection with soliton theory [14] and to the equivalence of the quantized SGE with the charged-zero sector of the massive Thirring model [15]. So studies of the interactive property dromions for this model are significant. It is well known that the interactions of (1+1)-dimensional solitons are elastic. There is no exchange of energy (no change of shape and velocity) among interacting solitons. But different results have been reported for (2+1)-dimensional integrable systems. For example, the dromion interactions reported are inelastic for the DS equation [16], but for modified KdV (MKdV) and Sawada-Kotera (SK) systems the interactions are elastic [17]. The question why the interaction between dromions is elastic for some models and inelastic for others has not yet been answered and shall be dealt with in this paper. We also hope to learn whether there are different interactive properties when dromions are interacting due to the different selection of parameters for the same (2+1)-dimensional integrable model. In order to answer these questions, we study (2+1)-dimensional integrable SGE's with physical significance in detail.

The paper is organized as follows. In Sect. II, the multi-dromion solutions are given for (2+1)-dimensional integrable sine-Gordon (SG) systems. Plots of the interaction of two dromions for the SGE are shown in Section III. Section IV contains a summary and discussion.

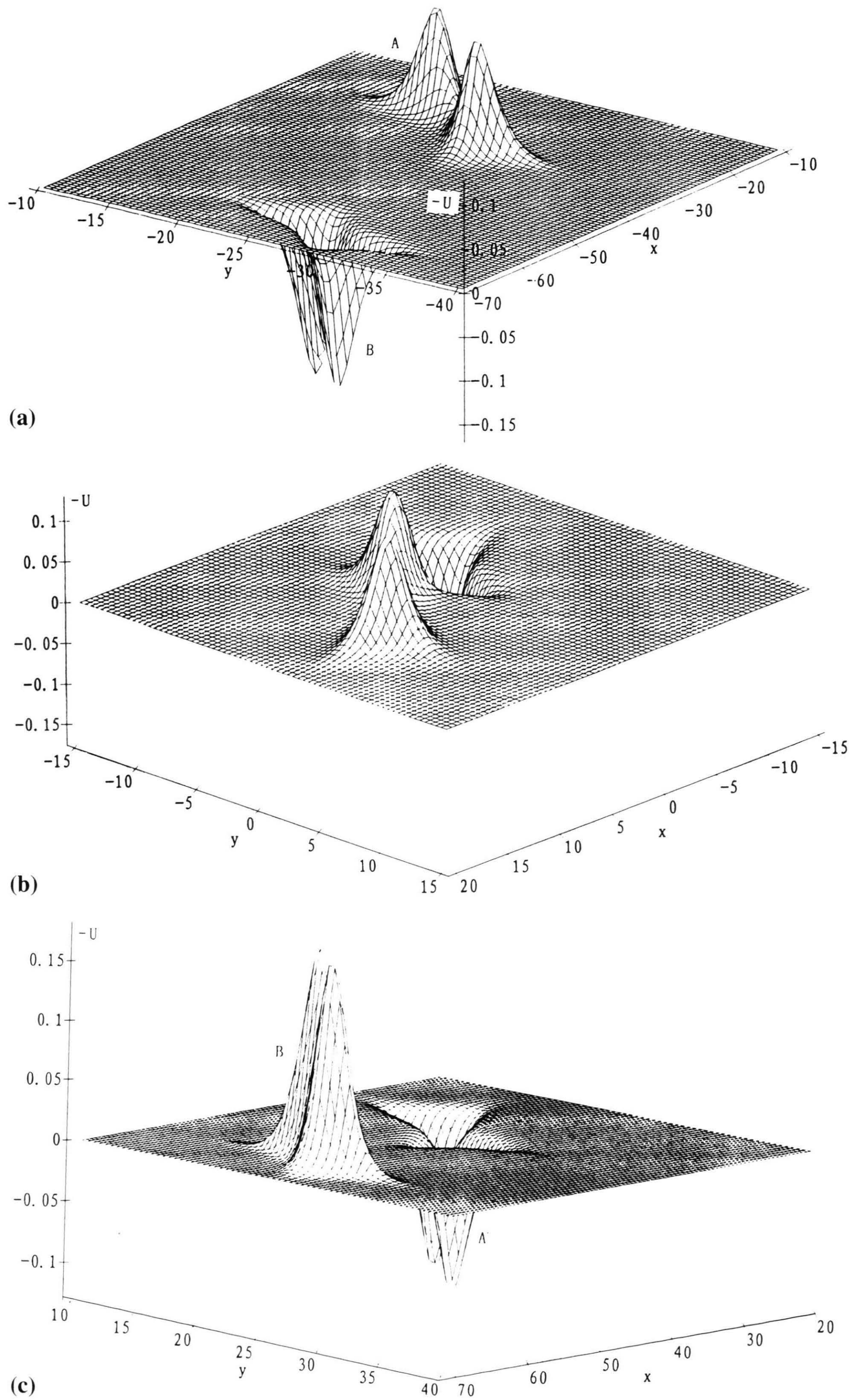
2. Multi-dromion Solutions of the (2+1)-dimensional sine-Gordon Equation

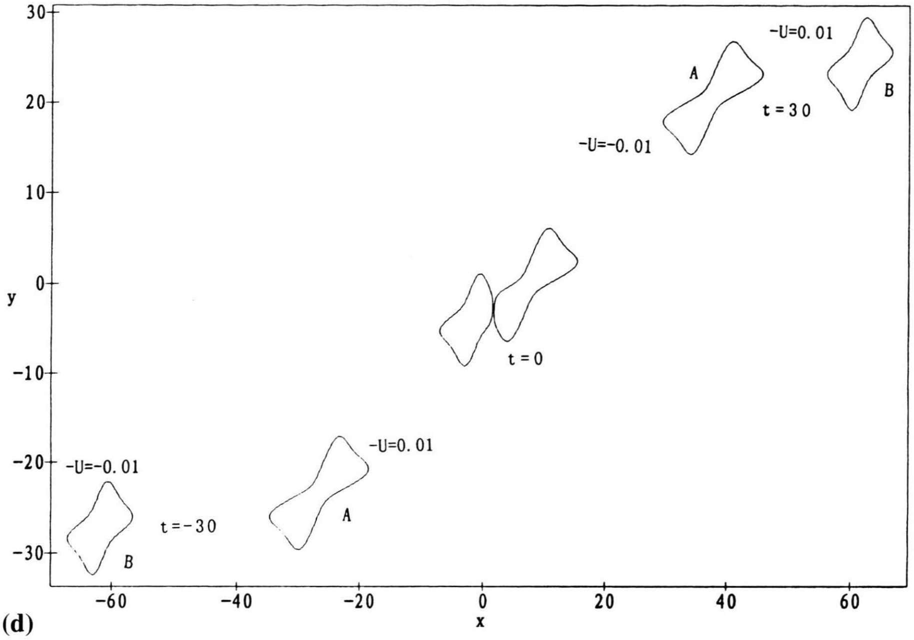
The bilinear form of a (2+1)-dimensional SG system can be written as

$$A(D_X)\,(f\cdot f-g\cdot g)\equiv A(D_x,D_y,D_t)\,(f\cdot f-g\cdot g)=0,\quad (1)$$

$$B(D_X)\,f\cdot g\equiv B(D_x,D_y,D_t)\,f\cdot g=0,\quad (2)$$

0932-0784 / 99 / 0500-0291 $ 06.00 © Verlag der Zeitschrift für Naturforschung, Tübingen · www.znaturforsch.com


A
B
-U
x
y
(a)
-U
x
y
(b)
-U
B
A
x
y
(c)



(d)

Fig. 1. The plots of the interaction of two dromions formed by three line solitons which are characterized by $\eta_1=x-1/2\,y-2\,t=x_1-2\,t$, $\eta_2=2/3\,x+y-27/35\,t=y_1-27/35\,t$, $\eta_3=-7/9\,x-2/3\,y+729/829\,t=-7/9\,x_1+729/829\,t$. The time of the figures reads: (a) $t=-30$, (b) $t=0$, (c) $t=30$. (d) is a cross section plot ($u=\pm 0.01$) in correspondence with (a), (b), and (c).

where A and B are even and odd functions of the variables $X=(x, y, t)$, $D_X=(D_x, D_y, D_t)$, and the D-operators are defined by [18, 19]

$$D_x^n D_y^m D_t^p f\cdot g \equiv (\partial_x-\partial_{x'})^n (\partial_y-\partial_{y'})^m (\partial_t-\partial_{t'})^p \cdot (f(X)\cdot g(X'))_{X'=X}\,. \tag{3}$$

It may be proved that a single dromion solution of the equations (1, 2) exists if a physical field

$$u=L(\partial_X)\,K(\partial_X)\,(\tan^{-1}(g/f)) \equiv (a_1\,\partial_x+b_1\,\partial_y)\,(a_1\,\partial_x+b_2\,\partial_y)\,(\tan^{-1}(g/f)) \tag{4}$$

is defined, where a_1, b_1, a_2, b_2, should be selected such that the linear operators $L(\partial_X)$, $K(\partial_X)$ annihilate two line solitons

$$f=1+a_{12}\exp(\eta_1+\eta_2),\; g=\exp(\eta_1)+\exp(\eta_2), \tag{5}$$

$$\eta_i=p_i\,x+q_i\,y+w_i\,t+\text{const}\equiv P_i\cdot X+\text{const} \tag{6}$$

with

$$P_i=(p_i, q_i, w_i),\quad (i=1,2),\quad B(P_i)=0$$

$$a_{12}=\frac{A(P_1-P_2)}{A(P_1+P_2)}\,. \tag{7}$$

That is to say, in the space-time (x, y, t) the dromion solutions are driven by ghost line solitons, which are nonparallel to each other. Two line solitons are annihilated by two linear operators $L(\partial_X)$ and $K(\partial_X)$, while a dromion, which is located at the crossing point of the two line solitons survives. From (4)–(6) we see that, if we take a space transformation

$$p_1\,x+q_1\,y=p\,x_1,\quad p_2\,x+q_2\,y=q\,y_1,$$

$$\Delta\equiv p_1\,q_2-p_2\,q_1\neq 0 \tag{8}$$

and fix the constants a_i, b_i, in (4) as $a_1=-q_1\,q/\Delta$, $b_1=p_1\,q/\Delta$, $a_2=q_2\,p/\Delta$, $b_2=-p_2\,p/\Delta$, then (4)–(6) are changed to

$$u=(a_1\,\partial_x+b_1\,\partial_y)\,(a_2\,\partial_x+b_2\,\partial_y)\,(\tan^{-1}(g/f)) \equiv \partial_{x_1}\partial_{y_1}(\tan^{-1}(g/f)), \tag{9}$$

$$f=1+a_{12}\exp(\eta_1+\eta_2),\; g=\exp(\eta_1)+\exp(\eta_2),$$

$$\eta_1=p\,x_1+\text{const},\; \eta_2=q\,y_1+\text{const}. \tag{10}$$

Concretely, we discuss the dromion structures for the following (2+1)-dimensional integrable SGE [20]

$$A(D_X)\,(f\cdot f-g\cdot g)=D_x\,D_t\,(f\cdot f-g\cdot g)=0, \tag{11}$$

$$B(D_X)\,f\cdot g=(D_x^3\,D_t+D_y\,D_t+a)\,f\cdot g=0\,. \tag{12}$$

Using the general step developed by Hirota, the N line soliton solution of the equation system (11, 12) can be written as

$$f(x,y,t)=\sum_{n=0}^{N/2}\sum_{{}_N C_{2n}} a(i_1,i_2,\ldots,i_{2n})\exp(\eta_{i1}+\eta_{i2}+\ldots+\eta_{i2n}), \quad (13)$$

$$g(x,y,t)=\sum_{m=0}^{[(N-1)/2]}\sum_{{}_N C_{2m+1}} a(j_1,j_2,\ldots,j_{2m+1})\exp(\eta_{j1}+\eta_{j2}+\ldots+\eta_{j2m+1}), \quad (14)$$

$$a(i_1,i_2,\ldots,i_n)=\begin{cases}\Pi_{k,l}^{(n)}\, a(i_k,i_l) & \text{for } n\geq 2\\ 1 & n=0,1\end{cases}, \quad (15)$$

$$a(i_k,i_l)=\frac{A(P_{ik}-P_{il})}{A(P_{ik}+P_{il})}=\frac{(p_{ik}-p_{il})(w_{ik}-w_{il})}{(p_{ik}+p_{il})(w_{ik}+w_{il})}, \quad (16)$$

$$\eta_i=p_i x+q_i y+w_i t+\eta_{i0}, \quad (17)$$

$$B(P_{ik})=p_i^3 w_i+q_i w_i+a=0. \quad (18)$$

$[N/2]$ denotes the maximum integer which does not exceed $N/2$ and n_{i0} is an arbitrary but finite real constant related to the phase of the i-th soliton. ${}_N C_n$ indicates summation over all possible combinations of n elements taken from N, and $\Pi_{i,l}^{(n)}$ indicates the product of all possible combinations of the n elements. From (9) and (10) we know, because a_i, b_i are q_i, p_i dependent, that multi-dromion solutions for the potential u given by (10) are allowed only for a special form such that two linear operators $a_i\,\partial_x+b_i\,\partial_y (i=1,2)$ with fixed a_i b_i annihilate all the line solitons. So the multi-dromion solutions exist only for the potential form in the new space coordinates x_1, y_1

$$u=\partial_{x_1}\partial_{y_1}(\tan^{-1}(g(x_1,y_1,t)/f(x_1,y_1,t)), \quad (19)$$

where the forms of $g(x_1,y_1,t)$ and $f(x_1,y_1,t)$ are the same as those of (13) and (14), but η_i should be taken as

$$\eta_i=p_i x+q_i y+w_i t+\eta_{i0}=p_i' x_1+w_i t+\eta_{i0} \quad \text{or}$$
$$\eta_i=p_i x+q_i y+w_i t+\eta_{i0}=q_i' y_1+w_i t+\eta_{i0}. \quad (20)$$

As an example, we write down the explicit forms of f and g for $N=3$:

$$f(x,y,t)=1+a(1,2)\exp(\eta_1+\eta_2)+a(1,3)\exp(\eta_1+\eta_3)+a(2,3)\exp(\eta_2+\eta_3), \quad (21)$$

$$g(x,y,t)=\exp(\eta_1)+\exp(\eta_2)+\exp(\eta_3)+a(1,2)\,a(1,3)\,a(2,3)\exp(\eta_1+\eta_2+\eta_3), \quad (22)$$

$$\eta_i=p_i' x+w_i t+\eta_{i0} \quad \text{or} \quad \eta_i=q_i' y+w_i t+\eta_{i0}, \quad (23)$$

$$p_i^3 w_i+q_i w_i+a=0, \quad (24)$$

$$a(i,j)=\frac{(p_i-p_j)(w_i-w_j)}{(p_i+p_j)(w_i+w_j)}. \quad (25)$$

3. Dromion Interactions

It is known that in (1+1)-dimensions there is no exchange of physical quantities like energy and momentum of the solitons after collision. Except for the phase shifts, the velocities and shapes remain unchanged.

We hope to learn whether a similar property is valid or not for the interactions among dromions for (2+1)-dimensional integrable models. Especially, we hope to learn whether the interactions are parameter dependent or not.

It is difficult to study the interaction of the dromions analytically because of the complexity of the multi-dromion solutions. It is more straightforward to study the dromion interactions graphically.

Figure 1 is the interaction plot of two dromions which are formed by three ghost line solitons for the SGE, where the three ghost line solitons are characterized by

$$n_1=x-\frac{1}{2}y-2t=x_1-2t, \quad (26)$$

$$n_2=\frac{2}{3}x+y-\frac{27}{35}t=y_1-\frac{27}{35}t, \quad (27)$$

$$n_3=-\frac{7}{9}x-\frac{2}{3}y+\frac{729}{829}t=-\frac{7}{9}x_1+\frac{729}{829}t, \quad (28)$$

and $a(1,2)$, $a(1,3)$ and $a(2,3)$, determined by (p_i, q_i), are all nonzero in (21) and (22). In Figs. 1 (a), 1 (b), and 1 (c), the time t is taken as –30, 0, and 30, respectively. Figure 1 (d) is a cross section plot of the two dromions before and after interaction in correspondence with Figs. 1 (a), 1 (b), and 1 (c), respectively, while $u=\text{const}=\pm 0.01$. Comparing Figs. 1 (d) with Fig. 1 (a)–(c), one can clearly see that the shapes of the two dromions are totally equal but their directions have been exchanged when they are interacting, that means there is not exchange of energy and momentum but there is exchange of some kind of angular momentum, and the phase shifts.

Figure 2 is the interaction plot of two dromions, which are formed by three ghost line solitons characterized by

$$n_1=\frac{1}{2}x+\frac{2}{3}y-\frac{24}{19}t=\frac{1}{2}x_1-\frac{24}{19}t, \quad (29)$$

$$n_2=\left(\frac{2}{3}+\left(\frac{1}{2}\right)^3-\frac{3}{4}\right)^{1/3}x+\frac{3}{4}y-\frac{24}{19}t=\frac{3}{4}y_1-\frac{24}{19}t, \quad (30)$$

$$n_3=\frac{1}{3}x+y-\frac{27}{28}t=\frac{1}{3}x_1-\frac{27}{28}t. \quad (31)$$

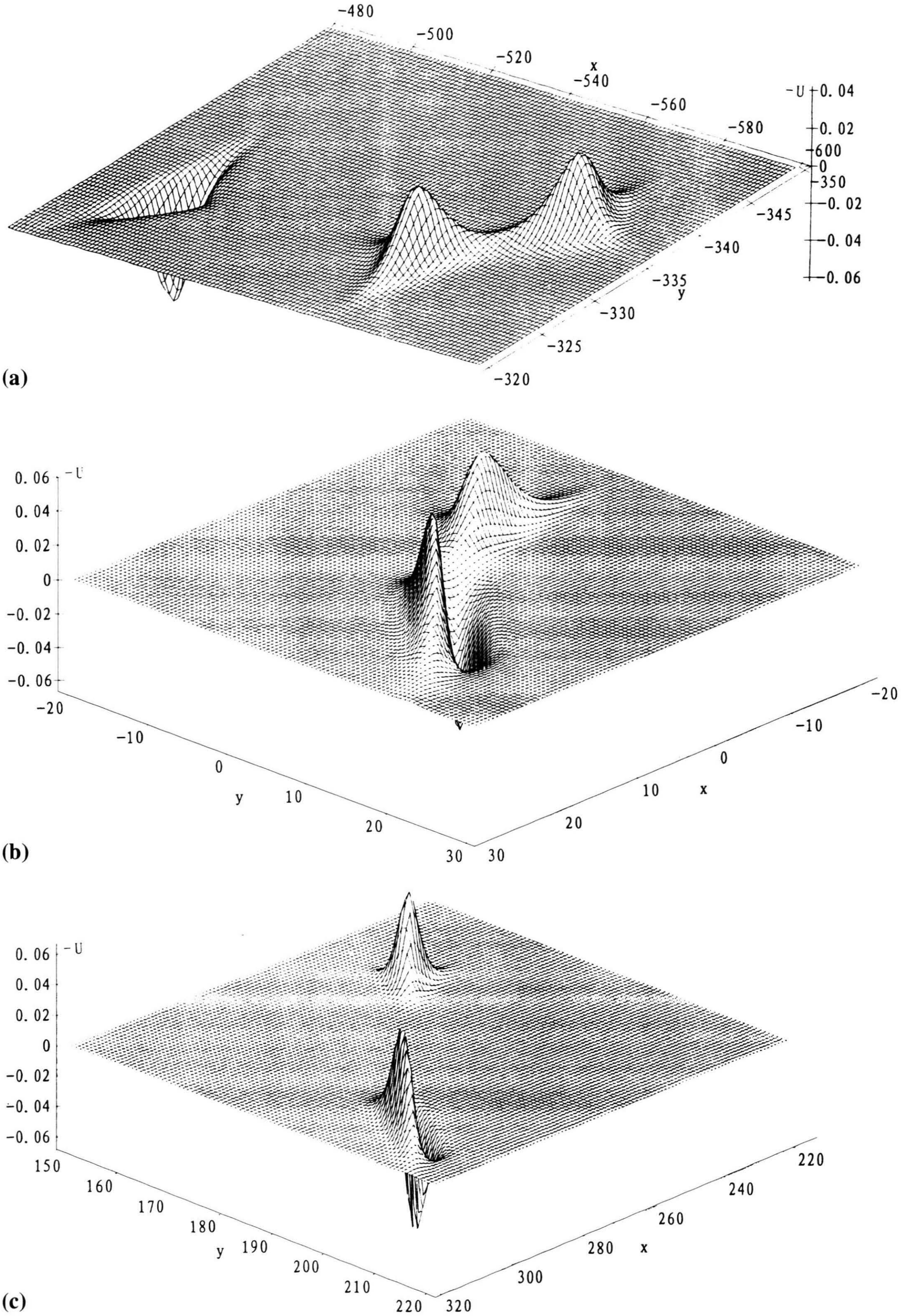


Fig. 2. The plots of the interaction of two dromions for this model. The related three line solitons are determined by $\eta_1 = 1/2\,x + 2/3\,y - 24/19\,t = 1/2\,x_1 - 24/19\,t$, $\eta_2 = (2/3 + (1/2)^3 - 3/4)^{1/3}x + 3/4\,y - 24/19\,t = 3/4\,y_1 - 24/19\,t$, $\eta_3 = 1/3\,x + y - 27/28\,t = 1/3\,x_1 - 27/28\,t$. The time of the figures reads: (a) $t = -200$, (b) $t = 0$, (c) $t = 100$.

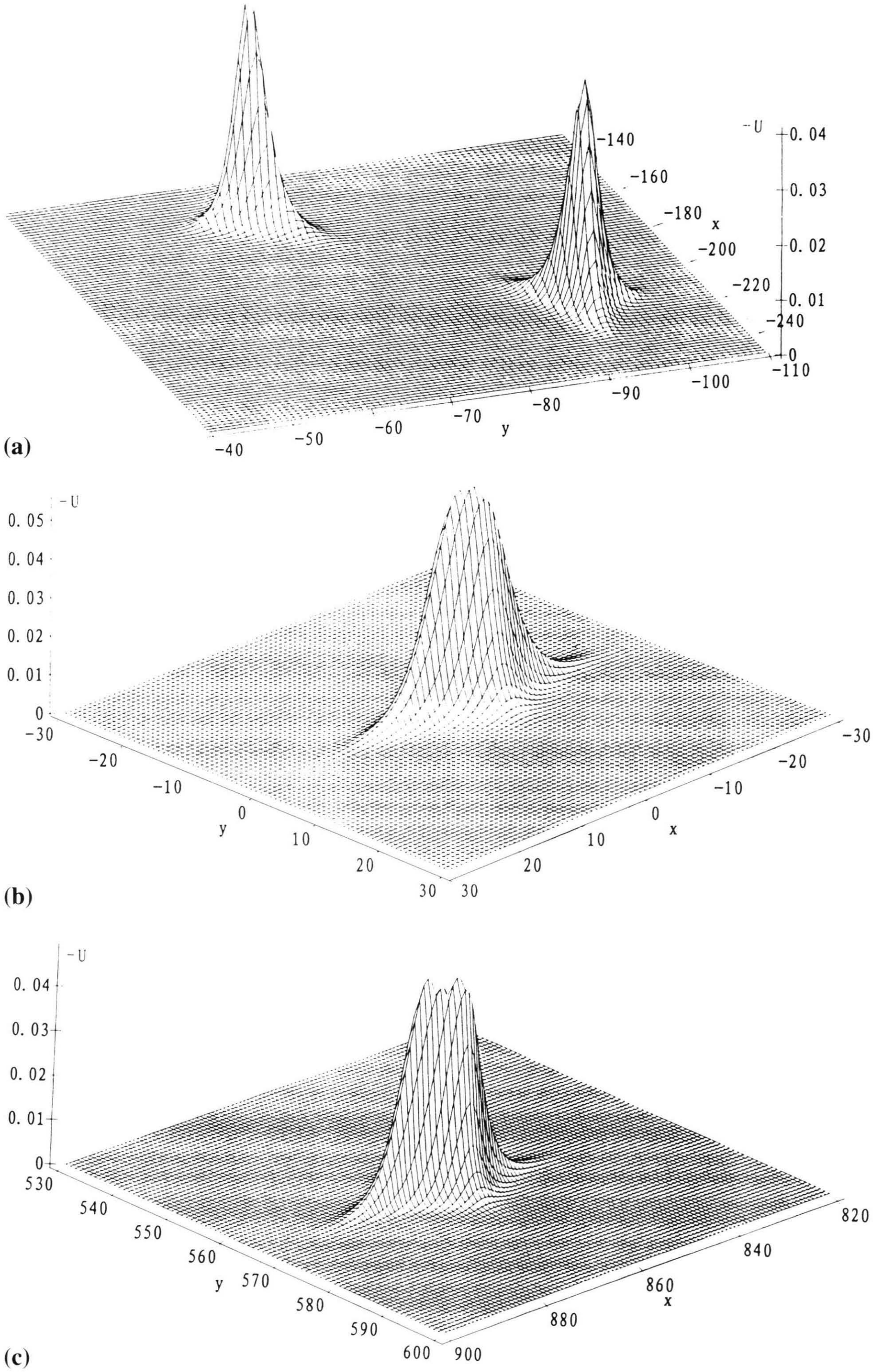


Fig. 3. The plots of the interaction of two dromions for the same equation. The corresponding three line solitons are determined by $\eta_1 = 1/3x + 2/3y - 27/19t = 1/3x_1 - 27/19t$, $\eta_2 = (2/3 + (1/3)^3 - 3/4)^{1/3}x + 3/4y - 27/19t = 3/4y_1 - 27/19t$, $\eta_3 = 1/3x + y - 27/28t = 1/3x_1 - 27/28t$. The time of the figures reads: (a) $t = -50$, (b) $t = 0$, (c) $t = 300$.

Because of the parameter-values in (29)–(31), $a\,(1,2)=0$, $a\,(1,3)\neq 0$, $a\,(2,3)\neq 0$ in (21) and (22). In the Figs. 2 (a), 2 (b), and 2 (c), the time t is taken as –200, 0, and 100, respectively. Not as in Fig. 1, from Figs. (2a)–(2b) we can clearly see that the shapes of the two dromions are changed after interaction. That is to say that there is exchange of energy and momentum between the dromions when they are interacting.

Figure 3 is the interaction plot of two dromions which are formed by three ghost line solitons for the same model, but the three ghost line solitons are determined by

$$n_1=\frac{1}{3}x+\frac{2}{3}y-\frac{27}{19}t=\frac{1}{3}x_1-\frac{27}{19}t, \tag{32}$$

$$n_2=\left(\frac{2}{3}+\left(\frac{1}{3}\right)^3-\frac{3}{4}\right)^{1/3}x+\frac{3}{4}y-\frac{27}{19}t=\frac{3}{4}y_1-\frac{27}{19}t, \tag{33}$$

$$n_3=\frac{1}{3}x+y-\frac{27}{28}t=\frac{1}{3}x_1-\frac{27}{28}t, \tag{34}$$

and $a\,(1,2)$ and $a\,(1,3)$ are zero while $a\,(2,3)\neq 0$ in the functions of f and g because of p_i and q_i in (23)–(25). In the Figs. 3 (a), 3 (b), and 3 (c), t is taken as –50, 0, and 300, respectively. From Fig. 3 one can see that two dromions before interaction become one dromion after interaction. That is to say, the collision between two dromions is completely inelastic.

4. Summary and Discussions

In summary, we have constructed multidromion solutions of the (2+1)-dimensional sine-Gordon type for some suitable potentials. The multidromions are constructed by multi-line solitons, say a single dromion is constructed by two line solitons and two dromions are constructed by three line solitons. All the line solitons should be parallel to the new axes $\{x_1, y_1\}$.

For (1+1)-dimensional integrable models, like the KdV equation, the interaction among solitons is completely elastic. There is no energy and momentum exchange among solitons when they are interacting. The only effect of the soliton interaction is phase shifts. However, for the (2+1)-dimensional SGE there result different interaction properties for the same model because of different parameters (p_i, q_i). If all the coupling coefficients (interaction constants) a_{ij} are non zero, the interaction between two dromions is completely elastic (there is no exchange of energy and momentum except for exchange of some special kinds of angular momentum and the phase shift). This interaction is similar to that in the (1+1)-dimensional models. If one of the three coupling coefficients is zero, the shape of two dromions before and after interaction will be changed. That means the interaction of dromions in this case is not completely elastic (there is exchange of energy and momentum). If two of the three coupling coefficients are zero, two dromions will become one dromion after interaction. So this kind of interaction between dromions is completely inelastic.

We have given pictures of interaction for (2+1)-dimensional integrable SGE's. There are three different interaction behaviours between dromions in one model. The conclusions given above are similar to collisions of classical particles. It is known that the collision between two classical particles may be elastic, not completely elastic, or completely inelastic according to the material properties. In this paper we show that also the interaction between two dromions may be elastic, not completely elastic, or completely inelastic according to whether the coupling coefficients equal zero or don't. Whether these phenomena can take place in other (2+1)-dimensional integrable models or in (1+1)-dimensional integrable models is worth to be further studied.

The work was supported by the National Natural Science Foundation of China. We would like to thank Prof. S.-y. Lou for helpful discussions.

[1] M. Boiti, J. J. P. Leon, L. M. Martina, and F. Pempinelli, Phys. Lett. **A132,** 432 (1988).
[2] A Davey and K. Stewartson, Proc. Roy. Soc. London **A360,** 592 (1978).
[3] L. P. Nizhnik, Sov. Phys. Dokl, **25,** 707 (1980); A. P. Veselov and S. P. Novikov, Sov. Math. Dokl. **30,** 588, 705 (1984); S. P. Novikov and A. P. Veselov, Physic **D18,** 267 (1986).
[4] R. Radha and M. Lakshmanan, J. Math. Phys. **35,** 4746 (1994).
[5] J. Hietarintr, Phys. Lett. **149A,** 133 (1990).
[6] M. Boiti, J. J. p, M. Manna, and F. Pempinelli, Inverse Problems **2,** 271 (1986); **3,** 25 (1987).
[7] R. Radha and M. Lakshmanan, Phys. Lett. **A197,** 7 (1995).
[8] S.-y. Lou, J. Phys. A: Math. Gen. **28,** 7227 (1995).

[9] L. P. Eisenhart, Differential Geometry of Curves and Surfaces, Dover, New York 1960.

[10] A. J. Demaria, D. A. Stetser, and W. H. Glenn, Jr., Science **156,** 1557 (1967); G. Lamb, Jr., Phys. Lett. A **25,** 181 (1967); Rev. Mod. Phys. **43,** 99 (1971).

[11] H. Washimi and T. Taniuti, Phys. Rev. Lett. **17,** 996 (1966); F. D. Tappert, Phys. Fluids **15,** 2446 (1972).

[12] B. D. Josephson, Phys. Lett. **1,** 251 (1962); P. W. Anderson and J. M. Rowell, Phys. Rev. Lett. **10,** 230 (1963). B. D. Josephson, Adv. Phys. **14,** 419 (1965).

[13] R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **10,** 4114, 4130, 4138 (1974), and D **11,** 3424 (1975); R. Rajaraman, Phys. Rep. C **21,** 227 (1975).

[14] A. C. Scott, F. Y. F. Chu, and D. W. Mclaughlin, Proc. IEEE **61,** 1443 (1973), and references therein.

[15] S. Coleman, Phys. Rev. D **11,** 2088 (1975).

[16] J. Hieterinta and R. Hirota, Phys. Lett. **A 145,** 237 (1990).

[17] H.-y. Ruan and Y.-x. Chen, J. Math. Phys. **39,** (1998).

[18] R. Hirota, Phys. Rev. Lett. **27,** 1192 (1971).

[19] R. Hirota, Phys. Rev. Lett. **27,** 1192 (1971)

[20] K. M. Tamizhmani, A. Ramani, and B. Grammaticos, J. Math. Phys. **32,** 2635 (1991).